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Students' Understanding and Use of Representations with Vector Concepts

Introduction

As mathematics teachers, we have a growing concern in regards to how well our students understand the mathematical concepts in our curriculum. It seems that many of our students come through math classes with only a procedural understanding. We have noticed that when these students move on to math analysis it becomes evident that their understanding is limited to calculations. Research has shown that the expectations placed on the students by the teacher in a given class may drive most students to the required knowledge whether it is procedural or conceptual (Thompson A.G., Philipp, Thompson P.W., & Boyd 1994).

We had the opportunity to join a research project that involved eight students in a university linear algebra class. We will focus on four of the students and their responses from interviews seeking their understanding of vectors. Each case study will categorize their understanding as conceptual or procedural from their responses. Their personal concept definitions and concept images, including linear dependence, linear independence, and span, will be discussed, as well as their use of geometric and algebraic representations, and their transfer between such.

It was evident that some students had disjoint mental spaces, meaning they could not make relationships between the pieces of knowledge they held. Some students would rely on the theorems they had read in the textbook but could not effectively apply the theorems in situations where the theorems would be useful. We observed that some of the students' concept images contained ideas or relationships that were not close to the accepted formal concept definitions. Overall, we will note how the students' concept image, concept definition, and type of understanding they used affected their ability to give conceptual answers to the questions asked.

Theoretical framework

Much research can be found discussing conceptual and procedural knowledge. Richard Skemp referred to two types of understanding as *relational* understanding, "knowing what to do and why" and *instrumental* understanding, "rules without reason" (Skemp, 1976, p. 20). One of the problems Skemp (1976) pointed out with instrumental understanding is that it involves a "multiplicity of rules" (p. 21). Students who are learning instrumental mathematics rely on a set of rules and memorized steps with the goal to arrive at the right solution.

Skemp's instrumental and relational understanding parallels Hiebert's and Lefevre's definition of *conceptual* and *procedural* understanding. According to Hiebert and Lefevre (1986), a person who possesses conceptual knowledge is capable of linking pieces of information in a network of ideas and is able to make and understand connections between relationships and other pieces of information. They can reason through a situation from their own understanding, with the proper use of concept definitions, relations, and representations (Alagic, 2003). They can connect new information with previously learned information in meaningful ways and can make sense of what they know so they can remember and apply important mathematical concepts (Alagic, 2003).

On the other hand, Hiebert and Lefevre (1986) define procedural knowledge as being made up of two parts. "One part is composed of the formal language, or symbol representation system, of mathematics. The other part consists of the algorithms, or rules, for completing mathematical tasks" (p. 6). Hiebert and Lefevre (1986) stated, "We propose that procedures that are learned with meaning are procedures that are linked with conceptual knowledge" (p. 8). Furthermore, procedural learners tend to have a series of *disjoint mental spaces*, whereas the learners who exhibited a *single mental space* could connect ideas flexibly from chemistry to mathematics and vice versa (Simpson & Zakaria, 2004).

The orientation by which the students are instructed in a mathematics classroom influences how students will obtain their understanding of mathematical concepts. Thompson et al. (1994) refer to two main orientations as *conceptual* and *calculational.* The discussions in class are evidence of a teacher's orientation (Thompson et al., 1994). A teacher with a conceptual orientation has an image of a concept that is expressed "in ways that focus students' attention away from thoughtless application of procedures and toward a rich conception of situations, ideas and relationships among ideas" (Thompson et al., 1994, p. 86). Skemp (1976) noted four main advantages to teaching relational mathematics: "1) It is more adaptable to new tasks 2) It is easier to remember 3) Relational knowledge can be effective as a goal in itself 4) Relational schemas are organic in quality" (p. 23-24).

Calculational orientation of a teacher is evident by the teacher's "image of mathematics as the application of calculations and procedures for deriving numerical results" (Thompson et al., 1994, p. 86). With calculationally-oriented instruction learners focus on getting the correct answer and do not focus on their reasoning (Alagic, 2003). Thompson et al. also discussed a repercussion for students who are taught by a calculational orientation. Students who view "mathematics as answer-getting will not only have difficulty focusing on their and others' reasoning, they may also consider such a focus as being irrelevant to their images of what mathematics is about" (Thompson et al., 1994, p. 88).

Unfortunately, you see instrumental teaching and learning taking place because it is easier to understand, one can feel the rewards immediately, and one can usually get the correct answer quickly by following a memorized set of rules (Skemp, 1976). Students are most comfortable answering questions the way they were taught to answer them, so teachers should be teaching their students to think conceptually. When students arrive at an answer, they should know what the answer means and not just accept it because they followed the steps and calculations.

Two other terms that need to be noted are *concept definition* and *concept image*. Tall and Vinner (1981) define the concept definition as “a form of words used to specify that concept” (p. 152). There can be a formal definition that the mathematic world agrees upon or one’s own personal definition. The concept image is defined as “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (Tall & Vinner, 1981, p. 152). It is possible for a student to develop a concept image without connecting it to the concept definition or even give the correct concept definition upon request, yet have an inaccurate concept image (Tall & Vinner, 1981).

Students may also have difficulty seeing relationships of the same concept among multiple representations, which yields conflicts in their concept image. Alagic (2003) suggests that the students will often acquire a conceptual understanding if they can make the connections between the multiple representations. The different forms of representation are “useful tools” to construct and communicate information and understanding (Greeno et al., 1997, p. 362). When students encounter difficulties with disjointed mental spaces it may be due to the fact that they are not taught how to see the relationships or not expected to look for the connections.

A study done by Watson, Spyrou, & Tall (2003) focused on the concept of vector in the English curriculum. This research has shown that many sophisticated vector concepts can be reached when the focus is on the effect of an action. They refer to three worlds of mathematics: the *embodied world*, *proceptual* (both processes and concepts) *world*, and *formal world*. The embodied world contains perception and action, and it focuses on objects and their properties. The proceptual world has symbols that act as processes and concepts, and the formal world is of definitions and proof and focuses on properties and deductive relationships between them (Watson et al., 2003). Students who are successful in the proceptual world perform procedures and rely on and need the approval of a teacher. In the formal world the concept of vector is defined as "a vector space which consists of a set of elements (with no specific meaning except that they are called vectors) acted upon by a field of scalars satisfying certain properties" (Watson et al., 2003, Introduction section, para. 3).

Methods

Eight undergraduate students from the University of San Diego were interviewed for this research. These eight students were in an introductory linear algebra class and were chosen because they had demonstrated some success in their mathematics education. Each student was interviewed using a semi-structured task-based interview protocol, and the interviews lasted between 60-90 minutes. The student interviews were videotaped and any student work was collected. Christine Larson, Michael Smith, Chris Rasmussen, and Michelle Zandieh developed the questions.

In general, students were first asked for their definition of a vector followed by a question on a specific vector in $\mathbb{R}^3$ and the scalar multiples of that vector. After this, students were asked to show how they interpreted the specific vector geometrically. Next, students were asked about the span of the specific vector and the dimension of the span. They were then asked to give examples of vectors that were linearly dependent and linearly independent on the specific vector, and to describe algebraically and geometrically what it means for the vectors to be linear independent/dependent. One question had the students read the textbooks definition of what it means for two vectors to be linearly dependent and were asked to relate this definition to what they had described. The same line of questioning was then used for two specific vectors in $\mathbb{R}^3$. Last, the students were asked to describe the relationship between span, linear independence/dependence and dimension. Our concentration is on four students from the interviews: Rachel, Jason, Ryan, and Steven. Each will be discussed in the following four case studies in which excerpts are pulled from their interviews.

Findings

Case Study I

Robin was a student who was driven by procedures and wanted to solve for correct answers algebraically. She knew when to use the "echelon form thing." The following two excerpts will illustrate that she demonstrated that her concept image had disjoint mental spaces. Throughout the interview she frequently used terms such as "I guess," "maybe," "I don't know," "I kind of got confused," and "I did it wrong." She lacked confidence in most every response. Notice how her initial concept definition was graphically oriented.

EXCERPT 1

CHRISTY: How do you think about what a vector is?

ROBIN: Um, a line that goes one direction. So it starts at a point and then…It keeps going…

CHRISTY: I've heard another, another student in the class at one point say that they think of a vector as um a list of numbers. So what do you, what do you think of that that student's idea?

ROBIN: What do you mean a list of numbers? Just like, numbers? ... Well, I think of a vector as a line like. [Draws line in air] So then - - It's not a list of numbers.

CHRISTY: … like how would you write a vector?

ROBIN: Write a vector? Oh, I just think um like graphically – [CHRISTY: Uh-huh] a vector that it starts at a point and then just keeps going. [Crosses her index fingers, then places both fingers on the table with her right finger moving away from the fixed left finger] So it's not like necessarily like a list of numbers, [repeats same motion] it's just the numbers that the vector lies upon. I guess.

CHRISTY: Can you say that again. The - -

ROBIN: Like so say it's going like on a slope. [Holds arm in air at a slant] And then, um the list of numbers depends upon what the slope is rather than just any different numbers.

CHRISTY: So the numbers – So if you have some numbers that describe a vector they tell you about the direction.

ROBIN: Yeah.

Here it is evident that Robin's concept definition of a vector was conflicting with her concept image. Because she thought a vector kept going forever, it was then difficult for her to understand the other student's response as a list of numbers. She wanted to draw a line, keeping her discussion of vectors strictly graphical. Finally she allowed the numbers to be associated with a vector only as indicators of the direction of the vector. Throughout the interview, this personal concept definition she held influenced her responses to her graphical representations and questions on linear independence and span.

This next excerpt will highlight what little conceptual understanding Robin did have. Notice how she procedurally finds a linearly dependent vector. She knew it should be a multiple of the original, but when asked what that means geometrically she said, "Like if this is 2, 1, 0 then 4, 1, 0 would be like that on top of it." She did not catch what she had said and drew the vector directly on the original. With this idea that they lie on top of each other, she finds it difficult to see any connection to the algebraic idea of being a multiple.

EXCERPT 2

CHRISTY: Ok and why do you say that's linearly dependant on 2, 1, 0?

ROBIN: Because it's a multiple of it.

> CHRISTY: Ok, so can you tell me geometrically what it means for 2 linears, [ROBIN: That - -] two vectors to be linearly dependant?
>
> ROBIN: It lies together on - -
>
> CHRISTY: So, can you show me on … on here.
>
> ROBIN: Like, like if this is 2, 1, 0 then 4, 1, 0 would be like that on top of it, [draws another vector of same length to end of (2, 1, 0) on model]

Christy then asks Robin to read the textbook's definition of what it means algebraically for two vectors to be linearly dependent and then to explain how it relates to what she had drawn graphically.

> ROBIN: That's - - how do you relate that - - [CHRISTY: Ok.] Um, hmm [pause] that they're on top of each other?
>
> CHRISTY: [Laughs] I mean do you, do you, if you don't see any relationship that's fine.
>
> ROBIN: Yeah, I don't see any relationship…It's just a theorem, [CHRISTY: Ok] I guess.

Here Robin was quick to use her "rules without reason" and knew how to give an answer for a linear dependent vector. She had a difficult time when switching between representations. Knowing that "it's a multiple" did not allow her to see any relationship with her geometric vector. Without the knowledge of reason for her answers, besides the procedures to arrive at them, she could not express the relationship between the algebraic and geometric representations. As the concepts became more complex with span, linear independence, and dimension, Robin could not see any relationship between them. Robin seemed unaccustomed to answering conceptually driven questions.

Case Study II

Jason displayed mostly procedural knowledge. Throughout the interview he made several remarks about how he relied on the theorems and definitions from the book, and how he would try to relate these memorized theorems and definitions to what he was solving. Also, Jason made reference to his previous experience in learning mathematics and explained how he started to get better grades when he would memorize formulas and equations. This demonstrated Jason's resistance to think intuitively, because he did not have much success in mathematics when he relied on his own understanding. Jason needed to be asked to think intuitively about the questions and concepts. Jason had many missing concepts and incomplete schema on linear dependence, linear independence and span.

Jason did not have a connected relationship between his algebraic and geometric description of linear dependence, and relied on computation to validate the solution. His visual description of linear independence did not agree with his definition of linear dependence; Jason's concept image did not agree with his concept definition causing the conflicts in his transfer from the geometric to algebraic representations. Jason thought anytime you had a row of zeros in the row reduced echelon form it implied the problem at hand had a free variable, making the vectors involved linearly dependent. Jason did not understand that the row of zeros could mean that the vectors are in two dimensions or that you may have a situation where you are solving for the scalar values c1 and c2. In the excerpt Jason was asked if the vectors (2, 1, 0) and (0, 1, 0) are linearly independent and why.

EXCERPT 1

JASON: I can do the algebraic way or the visual way? [Laughs]

CHRISTY: Um. Let's go down the visual road.

JASON: O.K. … Again we got ah, x would be out 2 [Marks 2 on x-axis], y would be 1 [marks 1 on y-axis]. So if I go like this –

CHRISTY: Actually the 2, 1, 0 is the one you drew -- [JASON: Yeah.] earlier. So let's keep that one.

JASON: And 0, 1, 0 so 0, 1, 0 would be right here. [Draws (0, 1, 0)] [CHRISTY: Mm Hmm.] Are they linear independent? And I would say ah, yes.

CHRISTY: And why do you say that?

JASON: Because their only points of intersection are the origin.

CHRISTY: O.K. [Writes on her paper]

JASON: But then I would solve it in the matrix to see what the solution was going to be. . .

Then Jason does the row reduction steps and gets the matrix $\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$ and says. . .

JASON: Right now you know, right from the very start, that you have a free variable. So that tells you you'd be r linear, oops dependent, [Laughs] according to what my -- I told you they were earlier, independent -- Hmm – So my definition and my visual don't agree.

CHRISTY: Which one are you more convinced by?

JASON: Well this. [Points to paper]

CHRISTY: This one?

JASON: Yeah . . . Well just according to one of the theorems if they are multiples then they are linear dependent. Only 2 vectors – Of the 2 vectors – As far as visually seeing it – Um. I think again with this being a free variable [Points to last line of zeros in his work] you can make these anything [Points to row in matrix 1-1-0] so then maybe you're down to ah – Hmm. Well you have more variables then you do vectors. So then you can ah – so then you can make these [Points to rows 1-1-0, and 0-1-0] anything you want to obtain a solution. I don't know basically how to express that.

Jason did not show evidence of translating his knowledge between his algebraic and geometric representations. Even though in the beginning Jason answered the question correctly using the 3D model, he was not able to explain how his original geometric interpretation fit with his new conclusion based on his computations. Here is an example of a student finding validation in computation, as Watson et al. (2003) have explained will happen when a student is in the proceptual world. When it came down to it Jason believed his calculations over what he visually observed. In addition, even when he discussed what he saw visually he mentioned how he applied a theorem. He emphasized what he had read and memorized from the book. More importantly, he did not have a connected network of ideas relating to linear independence, and as a result was not aware of the ease in using the 3D Plexiglas model to answer the question. Jason's incomplete schema of linear independence was also evident in other parts of the interview where he was asked about the concepts of linear dependence and span.

Jason did not completely understand what it means for two vectors to be linearly independent or the idea of span of two linearly independent vectors, more specifically with (2, 1, 0) and (0, 1, 0).

EXCERPT 2

CHRISTY: Geometrically, what does it mean for two vectors say v1 and v2 to be linearly independent? [Reads off paper]

JASON: That means that they don't lie on the ah, same line or plane.

CHRISTY: Or plane? I think – I – I think I see what you mean when you say they don't lie on the same line. But, I'm not sure I know what you mean when you say they don't lie on the same plane. Can you either draw or show me what you mean by that?

JASON: Well, I think if you have 2 vectors, and they're not multiples of each other, then what they do, they form a plane between them. [Draws lines filling a triangular region on model between the vectors (2, 1, 0) and (0, 1, 0)] And anything – Ah – And then to be linearly independent it would have to be something sticking out right here. [Puts pen at (1, 1, 1)] It couldn't be on the plane. Then it would be dependent. Off the plane it would be independent. Anything –

…

CHRISTY: So are you – Tell me if I'm understanding you correctly. You're saying anything that's sort of in between here [Motions on model in between (2, 1, 0) and (0, 1, 0)] would be linearly dependent?

JASON: Yeah.

CHRISTY: And you're saying something out here [Puts vector at about (4, 0, 0)] would or would not?

JASON: That would be linear ah independent.

…

CHRISTY: And then do you view this [Points to triangular region of the xy plane he drew on model between the vectors (2, 1, 0) and (0, 1, 0)] as going on forever or do you, in your mind, does it stop at some point?

JASON: Um, I want to say it goes on forever but then my vector's points in here. [Points to the end of (2, 1, 0) he drew] But I think I read that the plane goes on for infinity.

The reason Jason did not fully understand span was because he was missing important concepts related the idea of planes in space and span. Jason did not understand that two linearly independent vectors span a plane because he did not comprehend that you could get the other vectors in the plane by taking scalar combinations of the two linearly independent vectors. Jason knew that any vector that lies in the space spanned by two linear independent vectors would be linearly dependent. Yet, Jason did not understand that the space spanned by two linearly independent vectors is a plane that extends forever. Jason saw the plane spanned by the two linearly independent vectors as only the space in between the two linearly independent vectors.

This lack of understanding prevented him from correctly answering the question he was asked about the space spanned by three linearly independent vectors. He described it as a 'wedge shape zone' that was also mirrored on the other side. Yet, it should be noted that Jason's thinking of span in three dimensions is on the right track, but his idea of span is limited to the space in between the linear independent vectors.

Case Study III

Ryan is a student who demonstrated conceptual knowledge, but still had some disjoint mental spaces most likely due to his lack of a formal concept definition upon which he heavily relied. It appeared that Ryan explicitly preferred procedural problems, which kept him from a connected understanding.

One reason why Ryan may have had disjoint mental spaces is because he, like Jason, relied heavily on the theorems from the book. He refers to the book several times. One theorem he referred to often will be exemplified in the following excerpt. He held onto this theorem as he attempted to answer questions about linear dependence and span. Notice that when he is asked for a number of vectors needed to span a space, he over qualifies the situation with his theorem. When he was confused he said, "I need to read in my books."

EXCERPT 1

CHRISTY: So first of all, can you give me a vector that's linearly dependent on 2, 1, 0…

RYAN: Ok, so it's gonna be like any scalar multiple, scalar multiple of 2, 1, 0. And if we had like, uh, one more vector like suppose uh 3, 1, 0 and one more vector. Ok, this is vector a [(2, 1, 0)], vector b [(3, 1, 0)], so we have vector c. So we have ok, 4, 5, 6 [writing out components for vector c]. We have d, 7, 8, 9. These are going to be linearly dependent because it's like uh 4 vectors and 3 entries. [CHRISTY: Ok.] So that's one of the theorem in the book, in the book.
…

CHRISTY: And do you see any of this as having any relationship with linear independence or with span?

RYAN: I think like the dimension is like the entries, right, in vector. So like if we have 2, 3. This is like a vector in R square. [CHRISTY: Mm hmm.] So we are like 3 vectors like this. [CHRISTY: Mm hmm.] That's like 1, 2, ok suppose 4, 5. [CHRISTY: Mm hmm.] Then the way I think is like uh, it's gonna be 2D vector. [CHRISTY: Mm hmm.] And uh, like we have 3 vectors, so these are going to be linearly dependent. [CHRISTY: Ok.] The way I think of the theorem, you know in the book. [CHRISTY: Mm hmm.] So 3 vectors, 2 entries, you know. So more entries then more vectors we have. Then they are going to be linearly - - I mean more vectors than entries are linearly dependent. Something. [CHRISTY: Ok.] Yeah.

…

CHRISTY: Yeah, say we had R3. Is there a minimum number of vectors you would need to span that?

RYAN: Um, like 4 vectors. With - -

CHRISTY: You would need 4 to span?

RYAN: 3 entries. I don't know.

CHRISTY: So, so here I'm, I would refer to use this as being 3 vectors with 2 entries in each vector. … So you're saying if you have, if you have 4 then they would be linearly dependent?

RYAN: 4 vectors then, yeah that's - - For 3 entries right? I mean like to span? [CHRISTY: Mm hmm.] Yeah, then you can pretty much sure that it's gonna be, they're all gonna be in same span.

CHRISTY: Now why do you say, um - - Why would you say that 3 might not be enough? Just for the - -

RYAN: Well, it, yeah, it can be enough. A minimum is - - If you say minimum, then yeah, it can be enough. You said minimum, so - -

CHRISTY: It can be enough?

RYAN: Yeah. But like they have to be linearly dependent with each other according to other theorems, you know?

Ryan portrayed his reliance on a formal definition whether or not it was relevant. This theorem was inhibiting him from a larger conceptual understanding. This definition also came to the forefront when discussing dimension. Another possible conflict in his concept image was how he viewed dimension as the number of entries in a vector. With limiting personal concept definitions, he was unable to make the connections between the key elements of vectors.

Not only did he desire the book for his answers when he was confused, but he also wanted problems to solve. "If you have a problem like a numerical problem - - I can do it, but like, but that's what I think because we don't have that kind of problem." He displayed discomfort with the conceptual questions requiring him to externalize his internal representations.

During the interview Ryan, without hesitancy, did the algebraic procedure to add vectors and plotted the vectors to verify his answer. This would imply some connection of his understanding between algebraic and graphical representations. He was not able to explain his idea, but felt some intuition that he was correct. However, as he attempted other alternatives and still hoping to find some validity, he gave in saying, “No it doesn’t relate.” He wanted to connect his ideas, but he had a concept image that was weak in at least one area.

Ryan seemed to be more concerned with saying the correct answer than answering the questions given. He always wanted to just do a typical problem where he felt successful. He felt he needed to justify his performance by saying, “You know, like if you don’t have a problem. It’s like sometime it’s hard to explain the concept - - You know a concept because you can’t explain it sometime.”

Case Study IV

Steven was a student who could recall and correctly use formulas and procedures in relation to linear dependent vectors, was able to come up with examples of linear dependent vectors using the 3D model, and could describe linear dependence both algebraically and geometrically. It was evident that his concept definition of linear dependence was close to the formal concept definition. Although Steven could connect ideas relating to linear dependence, his concept image of span and dimension did not fit with the formal theory relating these two ideas.

Steven could describe geometrically and algebraically what it meant for two vectors to be linearly dependent. In addition, when asked to relate the textbooks algebraic description for what it means for two vectors to be linearly dependent to his geometric description, he was able to use the model to help relate these ideas. He came up with an example to explain linear dependence. In addition, he said that the books equation for two vectors to be linearly dependent $c_1v_1 + c_2v_2 = 0$ is similar to the formula he came up with $av_1 = v_2$. Steven said you could move the term v_2 to the other side and put a constant in front of it, and wrote $a_1v_1 - (-1)a_2v_2 = 0$. Steven was able to quickly come up with an example (although in his example $v_1 = v_2$) and could manipulate his formula to relate to the textbook algebraic description of two vectors that are linearly dependent.

The model seemed to help Steven when he was asked to give an example of a third vector that would be linearly dependent on (2, 1, 0) and (0, 1, 0). He began with procedures and manipulating formulas.

EXCERPT 1

> STEVEN: So I would, I would take the first vector and cross it by the other vector and when I, when I take the cross product of them I would get a vector 0, 0, 2 [Writes on paper] and this would be a normal vector. Which – basically if you have the – If the two vectors were on the xy-plane the normal would be the z-axis. [Puts pencil up to model to show z-axis] [CHRISTY: Mm Hmm.] And then I would just take another, another vector, like another point which would be on the same plane and would be perpendicular to, to this normal.

But, once he visualized the vectors on the 3D model he was able to give an example of a vector that was linearly dependent on (2, 1, 0) and (0, 1, 0).

> STEVEN: So then a vector that would be linearly dependent on those two would just be ah – I'd just say 3, 2, 0 would be linearly dependent.
> …
> CHRISTY: So how did you get that example?
>
> STEVEN: Ah once – Once I visualized it again – And I looked at it. [Holds vector up to z-axis] I realized that actually for this situation the xy-plane is the, is the plane that we're looking at and since, so any vector that would fall in the span of the xy-plane would be linearly dependent with these other two vectors.

Steven was also able to connect his concept definition of span to linear dependence when he explained his reasoning.

It was evident throughout Steven's interview that his concept definition of dimension and span was linked in a relationship that is different from the formal concept definition. He had connected the concepts of dimension and span in an incorrect way, and because of this he struggled to correctly understand and explain the concepts and relationships between them. Steven thought the dimension of the span would be one less than the dimension of the vector. Steven first displayed his misunderstanding when he was asked how he thought about the span of the vector (2, 1, 0).

EXCERPT 2

> CHRISTY: How do, how do you think about the dimension of the span of 2, 1, 0?
>
> STEVEN: I think of-- Like the dimension is just like how many-- How many items there are in a vector would be the dimension. So this would be [Points to (2, 1, 0) on paper] like three items 3D, which would be in $\mathbb{R}_3$. If there were two items in the vector it would be in it would be 2D and it would be in $\mathbb{R}_2$. [CHRISTY: O.K.] And then for the span, I just think of it as being one less than the dimension of the vector is what the span is. So for example, here [Points to (2, 1, 0) on paper] I have three items so the span of this would be in—The span of this would be in 2D because that's one less than the number of dimensions in the vector. [Long pause]

His first intuition was to describe the span as one dimension less that the dimension of the vector. Then, after considering the span of two vectors and realizing the span of two vectors would be a plane, he changed his idea and said the span of (2, 1, 0) would be a line.

> STEVEN: So I guess—If you only had one—I guess what I said earlier, if you only had one vector though, then the span could only be on the line because you don't have—You need two vectors to de-, define the plane.

It was evident from this conversation that Steven was building his concept image through reasoning.

Steven was able to work with different representations of linear dependence. His concept image of span and dimension caused some possible conflicts with creating a single mental space for these vector concepts. However, he did exemplify his ability to reason which helped him to make some new connections in his disjoint mental space.

Conclusion

Through the process of analyzing these four students we conclude that these students were mostly procedurally driven. Some had strong concept definitions and/or concept images for isolated parts of vectors leading to disjoint mental spaces. This became evident due to the conceptual nature of the questions that the students had difficulty answering. They seemed to depend on the memorized facts and procedures. As Alagic (2003) noted, “Students who memorize facts or procedures without understanding often are not sure when or how to use what they know and such learning is often quite fragile” (p. 385). Using procedures had given them success in previous mathematical courses, but they were using “rules without reason” (Skemp, 1976, p. 20). Also, they used the procedure or representation that was most familiar to them or their current class. It is suggested that the students’ first method is to solve algebraically, when they could have arrived at a solution using a more efficient method (Knuth, 2000).

We also conclude that these students found validation to their reasoning from their textbooks, theorems, and algorithms. Greeno et al. (1997) noted, “if they learn to give only the answers and explanations that are specified by teachers and textbooks, they are likely to learn the practices of memorizing” (p. 362). Because of this memorization of procedures they lacked the meaning of their answers. “They will not only have difficulty focusing on both their and others' reasoning, they may also consider such a focus as being irrelevant to their images of what mathematics is about” (Alagic, 2003, p. 385). This altered image of mathematics kept them from a complete and connected understanding of concepts. In other words, they had disjoint mental spaces and there were conflicts with their concept images. These students were unable to externally represent connections between vector concepts. As Skemp (1976) says, "There is no awareness of the overall relationship between successive stages, and the final goal. And in both cases, the learner is dependent on outside guidance for learning each new 'way to get there'" (p. 25).

It was also evident that these students lacked familiarity with transfer between representations. Their concept images did not connect the concept definition to the various representations of vectors, span, linear dependence, and dimension. They compartmentalized their understanding of different vector concepts to the different representations. Due to the unfamiliarity with the representations they also formed misconceptions, which were evident in their answers.

Most importantly these students demonstrated extreme difficulty reasoning with conceptual questions. Perhaps, as Watson et al. (2003) say those students who experience success with procedures grow accustomed to them. Because of their success, the challenging conceptual questions are not sought after. Perhaps they do not even know that a bigger picture exists because of their achievement using the procedures and actions. These students had yet to understand the connections, reasoning, and purpose of their answers. Once they change their focus to higher level thinking they will be able to gain a more conceptual understanding of mathematics and make connections between the concepts they learn and have learned creating a greater concept image. We must note that we do not downplay the importance of procedures so long as they are logically derived from the concepts being taught and the purpose for each step is clear.

As teachers, we will take to our profession the importance of teaching for conceptual understanding. We need to require our students to "speak with meaning" so that the students verbalize the concepts (Clark, Carlson, & Moore, 2007). This will help them to become formal reasoners. We also need to focus on understanding students' personal concept images and concept definitions. Teachers should pull students aside and informally interview them any chance they can to allow for an opportunity to gain insight into students' thinking. We need to know what our students are thinking so we can help them correct their possible misunderstandings and help build their mathematical schema. As Thompson et al. (1994) states that for teachers to be conceptually orientated and to "focus students' attention away from thoughtless application of procedures and toward a rich conception of situations," our actions should be driven by:

- an image of a system of ideas and ways of thinking that she intends the students to develop,
- an image of how these ideas and ways of thinking can develop,
- ideas about features of materials, activities, expositions, and students' engagement with them that can orient students' attention in productive ways,
- an expectation and insistence that students be intellectually engaged in tasks and activities. (p. 85)

In addition, we will use multiple external representations in our classes and also require our students to use them when problem solving in order to foster within students a more complete concept image. We want to avoid strictly using algorithms to teach problem solving so that we can foster a conceptual understanding with multiple representations. Furthermore, to nurture their conceptual understanding we will integrate the concepts we are teaching with other disciplines, primarily with the sciences. We will attempt to help the students build their internal representations and schemas into a holistic mathematical base.

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